Central Limit Theorem for Branching Random Walks in Random Environment

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Abstract

We consider branching random walks in d-dimensional integer lattice with time-space i.i.d. offspring distributions. When $d \geq 3$ and the fluctuation of the environment is well moderated by the random walk, we prove a central limit theorem for the density of the population, together with upper bounds for the density of the most populated site and the replica overlap. We also discuss the phase transition of this model in connection with directed polymers in random environment.

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1 Introduction

We consider particles in $\mathbb{Z}^d$, performing random walks and branching into independent copies at each step of the random walk. When a particle occupies a site $x \in \mathbb{Z}^d$ at time $t \in \mathbb{N} = \{0, 1, ..\}$, then, it moves to a randomly choosen adjacent site y at time $t + 1$ and is replaced by k new particles with probability $q_{t,x}(k)$ ($k \in \mathbb{N}$). We assume that the offspring distributions $q_{t,x} = (q_{t,x}(k))_{k\in\mathbb{N}}$ are i.i.d. in time t and space x. This model was investigated earlier in [2, 3], and we call it the branching random walks in random environment (BRWRE). See section 1.1 for a more precise definition.

An object of central interest in this model is the population $N_{t,x}$ of the particles at time-space $(t, x) \in \mathbb{N} \times \mathbb{Z}^d$, and the total population $N_t = \sum_{x\in\mathbb{Z}^d} N_{t,x}$ at time t. Due to the random environment, the population has much more fluctuation as compared with the non-random environment case, e.g.,[15, section 4.2]. This fluctuation results from "disastrous locations" in time-space, where the offspring distribution $q_{t,x}(k)$ happens to assign extremely high probability to small k's. Thanks to the random walk, on the other hand, some of the particles are lucky enough to elude those disastrous locations. Therefore, the spatial motion component of the model has the effect to moderate the fluctuation.

As is discussed above, the random environment intensifies the fluctuation of the population, while the spatial motion moderates it. As was observed earlier in [3, Theorem 4], these competing factors in the model give rise to a phase transition as follows. When the randomness of the offspring distribution is well moderated by that of the random walk, the growth of the total population is of the same order as its expectation with strictly positive probability. When, on the other hand, the randomness of the environment dominates, the total population grows strictly slower than its expectation almost surely. We also discuss this phase transition later in this article. Interestingly, this phase transition shares the same aspect with, or even explains, the localization/delocalization transition of directed polymers in random environment [8], and of parabolic Anderson model with time-space i.i.d potentials, e.g.,[5].

In this article, we mainly consider the case in which the fluctuation caused by the random environment is well moderated by the random walk. It is known that this is the case if $d \geq 3$ and the mean offspring is controlled by a sqare moment condition [3, Theorem 4]–see Theorem

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1.2.1 below. We prove a central limit theorem for the density of the population (Theorem 1.2.1, Corollary 1.2.2), together with upper bounds for the density of the most populated site and the replica overlap (Proposition 1.2.3). Our method here is based on sqare moment estimates. In section 3, we discuss the phase transition of BRWRE in connection with that of the directed polymers in random environment [8].

1.1 Branching random walks in random environment (BRWRE)

We start with some remarks on the usage of the notation in this paper. We write $\mathbb{N} = \{0, 1, 2, ...\}$, $\mathbb{N}^* = \{1, 2, ...\}$ and $\mathbb{Z} = \{\pm x \ ; \ x \in \mathbb{N}\}$. Let $(\Omega, \mathcal{F}, P)$ be a probability space, which is not necessarily the one we define by (1.3)–(1.4) later on. We write $P[X] = \int X \, dP$ and $P[X : A] = \int_A X \, dP$ for a r.v.(random variable) X and an event A.

We now define the model. Let $p(\cdot, \ \cdot)$ be a transition probability for a Markov chain with a countable state space Γ. To each $(t, x) \in \mathbb{N} \times \Gamma$, we associate a distribution

$$q_{t,x} = (q_{t,x}(k))_{k \in \mathbb{N}} \in [0, 1]^{\mathbb{N}}, \quad \sum_{k \in \mathbb{N}} q_{t,x}(k) = 1$$

on $\mathbb{N}$. Then, the branching random walk (BRW) with offspring distribution $q = (q_{t,x})_{(t,x) \in \mathbb{N} \times \Gamma}$ is described as the following dynamics:

- At time $t = 0$, there is one particle at the origin $x = 0$.
- Suppose that there are $N_{t,x}$ particles at each site $x \in \Gamma$ at time t. At time $t + 1$, the ν-th particle at a site x ($\nu = 1, .., N_{t,x}$) jumps to a site $y = X^{\nu}_{t,x}$ with probability $p(x, y)$ independently of each other. At arrival, it dies, leaving $K^{\nu}_{t,x}$ new particles there.

We formulate the above description more precisely. The following formulation is an analogue of [15, section 4.2], where non-random offspring distributions are considered. See also [3, section 5] for the random offspring case.

• *Spatial motion:* A particle at time-space location (t, x) is supposed to jump to some other location $(t + 1, y)$ and is replaced by its children there. Therefore, the spactial motion should be described by assignning destination of the each particle at each time-space location (t, x). So, we are guided to the following definition. We define the measurable space $(\Omega_X, \mathcal{F}_X)$ as the set $\Gamma^{\mathbb{N} \times \Gamma \times \mathbb{N}^*}$ with the product σ-field, and $\Omega_X \ni X \mapsto X^{\nu}_{t,x}$ for each $(t, x, \nu) \in \Gamma \times \mathbb{N} \times \mathbb{N}^*$ as the projection. We define $P_X \in \mathcal{P}(\Omega_X, \mathcal{F}_X)$ as the product measure such that

$$P_X(X^{\nu}_{t,x} = y) = p(x, y) \quad \text{for all } (t, x, \nu) \in \mathbb{N} \times \Gamma \times \mathbb{N}^* \text{ and } y \in \Gamma. \tag{1.1}$$

Here, we interpret $X^{\nu}_{t,x}$ as the position at time $t+1$ of the children born from the ν-th particle at time-space location (t, x).

• *Offspring distribution:* We set $\Omega_q = \mathcal{P}(\mathbb{N})^{\mathbb{N} \times \Gamma}$, where $\mathcal{P}(\mathbb{N})$ denotes the set of probability measures on $\mathbb{N}$:

$$\mathcal{P}(\mathbb{N}) = \{q = (q(k))_{k \in \mathbb{N}} \in [0, 1]^{\mathbb{N}} \ ; \ \sum_{k \in \mathbb{N}} q(k) = 1\}.$$

Thus, each $q \in \Omega_q$ is a function $(t, x) \mapsto q_{t,x} = (q_{t,x}(k))_{k \in \mathbb{N}}$ from $\mathbb{N} \times \Gamma$ to $\mathcal{P}(\mathbb{N})$. We interpret $q_{t,x}$ as the offspring distribution for each particle which occupies the time-space location (t, x). The set $\mathcal{P}(\mathbb{N})$ is equipped with the natural Borel σ-field induced from that of $[0, 1]^{\mathbb{N}}$. We denote by $\mathcal{F}_q$ the product σ-field on Ω_q.

We define the measurable space $(\Omega_K, \mathcal{F}_K)$ as the set $\mathbb{N}^{\mathbb{N}\times\Gamma\times\mathbb{N}^*}$ with the product σ-field, and $\Omega_K \ni K \mapsto K^\nu_{t,x}$ for each $(t,x,\nu) \in \mathbb{N}\times\Gamma\times\mathbb{N}^*$ as the projection. For each fixed $q \in \Omega_q$, we define $P^q_K \in \mathcal{P}(\Omega_K, \mathcal{F}_K)$ as the product measure such that

$$P^q_K(K^\nu_{t,x} = k) = q_{t,x}(k) \quad \text{for all } (x,t,\nu) \in \Gamma\times\mathbb{N}\times\mathbb{N}^* \text{ and } k \in \mathbb{N}. \tag{1.2}$$

We interpret $K^\nu_{t,x}$ as the number of the children born from the ν-th particle at time-space location (t,x).
We now define the branching random walk in random environment. We fix a product measure $Q \in \mathcal{P}(\Omega_q, \mathcal{F}_q)$, which describes the i.i.d. offspring distribution assigned to each time-space location. Finally, we define $(\Omega, \mathcal{F})$ by

$$\Omega = \Omega_X \times \Omega_K \times \Omega_q, \quad \mathcal{F} = \mathcal{F}_X \otimes \mathcal{F}_K \otimes \mathcal{F}_q, \tag{1.3}$$

and $P^q, P \in \mathcal{P}(\Omega, \mathcal{F})$ by

$$P^q = P_X \otimes P^q_K \otimes \delta_q, \quad P = \int Q(dq)P^q. \tag{1.4}$$

We denote by $N_{t,x}$ the population at time-space location $(t,x) \in \mathbb{N}\times\Gamma$, which is defined inductively by $N_{0,x} = \delta_{0,x}$ for $t = 0$, and

$$N_{t,x} = \sum_{y\in\Gamma} \sum_{\nu=1}^{N_{t-1,y}} \delta_x(X^\nu_{t-1,y}) K^\nu_{t-1,y} \tag{1.5}$$

for $t \geq 1$. The total population at time t is then given by

$$N_t = \sum_{x\in\Gamma} N_{t,x} = \sum_{y\in\Gamma} \sum_{\nu=1}^{N_{t-1,y}} K^\nu_{t-1,y}. \tag{1.6}$$

We remark that the total population is exactly the classical Galton-Watson process if $q_{t,x} \equiv q$, where $q \in \mathcal{P}(\mathbb{N})$ is non-random.On the other hand, if Γ is a singleton, then N_t is the polulation of the Smith-Wilkinson model [18].

For $p > 0$, we write

$$m^{(p)} = Q[m^{(p)}_{t,x}] \text{ with } m^{(p)}_{t,x} = \sum_{k\in\mathbb{N}} k^p q_{t,x}(k), \tag{1.7}$$

$$m = m^{(1)}. \tag{1.8}$$

Note that for $p \geq 1$,

$$m^p \leq Q[m^p_{t,x}] \leq m^{(p)}$$

by Hölder's inequality. We set

$$\overline{N}_{t,x} = N_{t,x}/m^t \text{ and } \overline{N}_t = N_t/m^t. \tag{1.9}$$

$\overline{N}_t = N_t/m^t$ is a martingale (Lemma 1.3.2 below), and therefore the following limit always exists:

$$\overline{N}_\infty = \lim_{t\to\infty} \overline{N}_t, \quad P\text{-a.s.} \tag{1.10}$$

1.2 Results

Before we state our results, we fix our notation for simple random walk.

• *The random walk:* $(\{S_t\}_{t\in\mathbb{N}}, P_S^x)$ is a simple random walk on the d-dimensional integer lattice $\mathbb{Z}^d$ starting from $x \in \mathbb{Z}^d$. More precisely, we let $(\Omega_S, \mathcal{F}_S)$ be the path space $(\mathbb{Z}^d)^{\mathbb{N}}$ with the cylindrical σ-field, and let $\Omega_S \ni S \mapsto S_t$, $t \in \mathbb{N}$ be the projection. We define $p : \mathbb{Z}^d \times \mathbb{Z}^d \mapsto \{0, \frac{1}{2d}\}$ by

$$p(x,y) = \begin{cases} \frac{1}{2d} & \text{if } |x-y| = 1, \\ 0 & \text{if } |x-y| \neq 1, \end{cases} \tag{1.11}$$

where $|x| = (|x_1|^2 + .. + |x_d|^2)^{1/2}$ for $x \in \mathbb{Z}^d$. We consider the unique probability measure P_S^x on $(\Omega_S, \mathcal{F}_S)$ such that $S_t - S_{t-1}$, $t = 1, 2, ..$ are independent and

$$P_S^x\{S_0 = x\} = 1, \quad P_S^x\{S_t - S_{t-1} = y\} = p(0,y), \quad \text{for } y \in \mathbb{Z}^d.$$

In the sequel, P_S^0 will be simply written by P_S. We define the return probability of the simple random walk:

$$\pi_d = P_S(S_t = 0 \ \text{ for some } t \geq 1). \tag{1.12}$$

As is well-known, $\pi_1 = \pi_2 = 1$, and $\pi_d < 1$ for $d \geq 3$.
To state our results, we assume that $\Gamma = \mathbb{Z}^d$ and that $p(\cdot, \ \cdot)$ is given by (1.11). Then, with the notation introduced by (1.5)–(1.10), we state:

Theorem 1.2.1 *Suppose that*

$$m > 1, \ \ m^{(2)} < \infty, \ \ \text{and} \ \ d \geq 3. \tag{1.13}$$

Then, the following are equivalent:

(a) $\dfrac{Q[m_{t,x}^2]}{m^2} < \dfrac{1}{\pi_d}$, *where $\pi_d \in (0,1)$ is defined by (1.12).*

(b) $\lim_{t\to\infty} \overline{N}_t = \overline{N}_\infty$ *in* $\mathbb{L}^2(P)$.

(c) $\lim_{t\to\infty} \sum_{x\in\mathbb{Z}^d} \overline{N}_{t,x} f\left(t^{-1/2}x\right) = \overline{N}_\infty \int_{\mathbb{R}^d} f g_1$ *in* $\mathbb{L}^2(P)$ *for all* $f \in C_b(\mathbb{R}^d)$*. Here and in what follows,*

$$g_t(x) = \left(\frac{d}{2\pi t}\right)^{d/2} e^{-\frac{d|x|^2}{2t}}, \quad t > 0, \tag{1.14}$$

$\int_{\mathbb{R}^d} f g_1$ *is the abbreviation for* $\int_{\mathbb{R}^d} f(x) g_1(x)\, dx$*, and* $C_b(\mathbb{R}^d)$ *denotes the set of bounded continuous functions on* $\mathbb{R}^d$.

Theorem 1.2.1(a) controls the randomness of the environment in terms of that of the random walk. Theorem 1.2.1(b) in particular implies that $P(\overline{N}_\infty > 0) > 0$, i.e., the growth of the total population is of the same order as its expectation with strictly positive probability. In contrast with this, we will see that the total population grows strictly slower than its expectation almost surely, if either $d = 1, 2$, or the environment is random enough (Corollary 3.3.2 below). It is easy to deduce from Theorem 1.2.1(c) the following:

Corollary 1.2.2 *Suppose that*

$$m > 1, \ \ m^{(2)} < \infty, \ \ d \geq 3, \ \ \text{and} \ \ \frac{Q[m_{t,x}^2]}{m^2} < \frac{1}{\pi_d}.$$

Then, $P(\overline{N}_\infty > 0) > 0$ *and*

$$\lim_{t\to\infty} P\left(\left|\frac{1}{N_t}\sum_{x\in\mathbb{Z}^d} N_{t,x} f\left(t^{-1/2}x\right) - \int_{\mathbb{R}^d} f g_1\right| \geq \varepsilon \middle| \overline{N}_\infty > 0\right) = 0$$

for all $\varepsilon > 0$ *and* $f \in C_b(\mathbb{R}^d)$.

Corollary 1.2.2 tells us that, as $t \nearrow \infty$, the density or the spatial distribution

$$\rho_{t,x} = \frac{N_{t,x}}{N_t}, \quad x \in \mathbb{Z}^d$$

of the population converges to the standard normal distribution, if it is properly scaled. Other interesting objects related to the density would be

$$\rho_t^* = \max_{x\in\mathbb{Z}^d} \rho_{t,x}, \quad \text{and } \mathcal{R}_t = \sum_{x\in\mathbb{Z}^d} \rho_{t,x}^2.$$

ρ_t^* is the density at the most populated site, while $\mathcal{R}_t$ is the probability that a given pair of particles at time t are at the same site. $\mathcal{R}_t$ can be thought of as the replica overlap, in analogy with the spin glass theory. Clearly, $(\rho_t^*)^2 \leq \mathcal{R}_t \leq \rho_t^*$. We use the method in this paper to show the following upper bound for $\mathcal{R}_t$:

Proposition 1.2.3 *Suppose that*

$$m > 1, \quad m^{(2)} < \infty, \quad d \geq 3, \quad \text{and} \quad \frac{Q[m_{t,x}^2]}{m^2} < \frac{1}{\pi_d}.$$

Then, $P(\overline{N}_\infty > 0) > 0$ *and*

$$\mathcal{R}_T = O(T^{-d/2}) \quad \text{in } P\left(\cdot|\overline{N}_\infty > 0\right)\text{-probability,}$$

i.e., the laws $P\left(T^{d/2}\mathcal{R}_T \in \cdot|\overline{N}_\infty > 0\right)$, $T \geq 1$ *are tight.*

Remarks: After the first version of this article was submitted, a couple of related results are obtained.

(1) Y. Hu and N. Yoshida [14] prove the following localization result, which is in contrast with Proposition 1.2.3 above: Suppose that $m^{(3)} < \infty$, $Q(m_{t,x} = m) \neq 1$, $Q(q_{t,x}(0) = 0) = 1$ and $P(\overline{N}_\infty = 0) = 1$. Then, there exists a non-random number $c \in (0, 1)$ such that

$$\overline{\lim_{t\nearrow\infty}} \mathcal{R}_t \geq c, \quad P\text{-a.s.}$$

(2) Y.Shiozawa [16] considers branching Brownian motion in random environment, which can be thought of as a natural continuous counterpart of the discrete model considered in this article. He proves Theorem 1.2.1–Proposition 1.2.3 for the continuous setting.

1.3 Some basic properties of $N_{t,x}$

Here again, we only assume that (S_t, P_S^x) is a Markov chain on a countable state space Γ and with the transition probability $p(\cdot,\ \cdot)$. We denote the t step transition probability by

$$p_t(x,y) = P_S^x(S_t = y). \tag{1.15}$$

Define $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and

$$\mathcal{F}_t = \sigma(X^{\cdot}_{s,\cdot}, K^{\cdot}_{s,\cdot}, q_{s,\cdot}\ ;\ s \le t-1) \quad t \ge 1. \tag{1.16}$$

This definition is natural, because the configuration of the particles up to time t is determined by the above $\mathcal{F}_t$. Note that $X^{\cdot}_{s,\cdot}, K^{\cdot}_{s,\cdot}, q_{s,\cdot},\ s \ge t$ are independent of $\mathcal{F}_t$.

Lemma 1.3.1 *For $t < T$,*

$$P^q[N_{T,x}|\mathcal{F}_t] = \sum_{y\in\Gamma} N_{t,y} P_S^y[\prod_{u=0}^{T-t-1} m_{t+u,S_u} : S_{T-t} = x]. \tag{1.17}$$

In particular,

$$P^q[N_{T,x}] = P_S^0[\prod_{u=0}^{T-1} m_{u,S_u} : S_T = x] \quad \textit{and} \quad P^q[N_T] = P_S^0[\prod_{u=0}^{T-1} m_{u,S_u}] \tag{1.18}$$

Proof: Let $A \in \mathcal{F}_t$ be arbitrary. Then,

$$P^q[N_{T,x} : A] = \sum_{x_{T-1}\in\Gamma} \sum_{\nu\ge 0} P^q[\delta_x(X^{\nu}_{x_{T-1},T-1}) K^{\nu}_{T-1,x_{T-1}} : N_{T-1,x_{T-1}} \ge \nu, A].$$

By the independence, each expectation in the above sum is equal to

$$\begin{aligned} & P_X[\delta_x(X^{\nu}_{T-1,x_{T-1}})] P_K^q[K^{\nu}_{T-1,x_{T-1}}] P^q[N_{T-1,x_{T-1}} \ge \nu, A] \\ = \;& p(x_{T-1}, x) m_{T-1,x_{T-1}} P^q[N_{T-1,x_{T-1}} \ge \nu, A]. \end{aligned}$$

Hence,

$$P^q[N_{T,x} : A] = \sum_{x_{T-1}\in\Gamma} P^q[N_{T-1,x_{T-1}} : A] m_{T-1,x_{T-1}} p(x_{T-1}, x).$$

By proceeding inductively, the right hand side is equal to

$$\begin{aligned} & \sum_{x_t, x_{t+1}, .., x_{T-1}\in\Gamma} P^q[N_{t,x_t} : A] \left(\prod_{u=t}^{T-1} m_{u,x_u}\right) \left(\prod_{u=t}^{T-2} p(x_u, x_{u+1})\right) p(x_{T-1}, x) \\ = & \sum_{x_t\in\Gamma} P^q[N_{t,x_t} : A] P_S^{x_t}[\prod_{u=0}^{T-t-1} m_{t+u,S_u} : S_{T-t} = x]. \end{aligned}$$

Hence we have (1.17). $\square$

Lemma 1.3.2 *$(\overline{N}_t, \mathcal{F}_t)_{t\ge 0}$ is a martingale on $(\Omega, \mathcal{F}, P)$. Similarly, $(P^q[\overline{N}_t], \mathcal{F}_{q,t})_{t\ge 0}$ is a martingale on $(\Omega_q, \mathcal{F}_{q,t}, Q)$, where $\mathcal{F}_{q,t}$ is a σ-field generated by $q(\cdot,\ s)$, $s \le t-1$.*

Proof: If $t < T$, then,

$$P[N_T|\mathcal{F}_t] = \sum_{x\in\Gamma} P[N_{T,x}|\mathcal{F}_t] \stackrel{(1.17)}{=} m^{T-t} \sum_{x\in\Gamma}\sum_{y\in\Gamma} N_{t,y} P_S^y[S_{T-t} = x] = m^{T-t} N_t.$$

$\square$

2 Proof of the results

2.1 Lemmas

We assume that $\Gamma = \mathbb{Z}^d$ and that $p(\cdot,\ \cdot)$ is given by (1.11) from here on.

Lemma 2.1.1

$$\begin{aligned} P[N_{T,x}N_{T,\widetilde{x}}] &= m^T P_S(S_T = x)\delta_{x,\widetilde{x}} \\ &\quad + cm^T \sum_{t=0}^{T-1} m^t P^{x,\widetilde{x}}_{S,\widetilde{S}} \left[\alpha^{\sum_{u=1}^t 1\{S_u = \widetilde{S}_u\}} : S_t = \widetilde{S}_t,\ S_T = 0\right], \end{aligned}$$

where $\alpha = \frac{Q[m_{t,x}^2]}{m^2}$ *and* $c = \dfrac{m^{(2)}}{m} - 1$.

Proof: We follow [2, Lemma 20]. $N_{t,x}N_{t,\widetilde{x}} = \sum_{y,\widetilde{y}} F_{y,\widetilde{y}}$, where

$$F_{y,\widetilde{y}} = \sum_{\nu=1}^{N_{t-1,y}} \sum_{\widetilde{\nu}=1}^{N_{t-1,\widetilde{y}}} K^{\nu}_{t-1,y} K^{\widetilde{\nu}}_{t-1,\widetilde{y}} \delta_x(X^{\nu}_{t-1,y}) \delta_{\widetilde{x}}(X^{\widetilde{\nu}}_{t-1,\widetilde{y}}).$$

(1) We first consider the expectation of $F_{y,\widetilde{y}}$ with $y \neq \widetilde{y}$. In this case, $K^{\nu}_{t-1,y}$ and $K^{\widetilde{\nu}}_{t-1,\widetilde{y}}$ are independent under $P(\cdot|\mathcal{F}_{t-1})$. Therefore, we have

$$P[F_{y,\widetilde{y}}|\mathcal{F}_{t-1}] = N_{t-1,y}N_{t-1,\widetilde{y}} m^2 p(y,x)p(\widetilde{y},\widetilde{x}).$$

(2) We turn to the expectation of $F_{y,\widetilde{y}}$ with $y = \widetilde{y}$. In this case, $\{K^{\nu}_{t-1,y}\}_{\nu=1}^{N_{t-1,y}}$ are independent under $P(\cdot|\widetilde{\mathcal{F}}_{t-1})$, where

$$\widetilde{\mathcal{F}}_{t-1} = \sigma(\mathcal{F}_{t-1}, (q_{t-1,x})_{x\in\mathbb{Z}^d}).$$

For $y = \widetilde{y}$ and $x = \widetilde{x}$, we have

$$P[F_{y,y}|\widetilde{\mathcal{F}}_{t-1}] = N_{t-1,y}(N_{t-1,y} - 1)m^2_{t-1,y} p(y,x)^2 + N_{t-1,y} m^{(2)}_{t-1,y} p(y,x).$$

The first and second terms on the right-hand-side come respectively from off-diagonal and diagonal terms in $F_{y,y}$.
For $y = \widetilde{y}$ and $x \neq \widetilde{x}$, we have no diagonal terms in $F_{y,y}$. Therefore,

$$P[F_{y,y}|\widetilde{\mathcal{F}}_{t-1}] = N_{t-1,y}(N_{t-1,y} - 1)m^2_{t-1,y} p(y,x)p(y,\widetilde{x}).$$

We now introduce the following notation:

$$N_{t,x,\widetilde{x}} = N_{t,x}N_{t,\widetilde{x}} - N_{t,x}\delta_{x,\widetilde{x}}.$$

From the considerations in (1) and (2) above, and from $P[N_{t,x}] = m^t p_t(0,x)$, we obtain

$$\begin{aligned} P[N_{t,x,\widetilde{x}}] &= \sum_{\substack{y,\widetilde{y} \\ y\neq\widetilde{y}}} m^2 P[N_{t-1,y,\widetilde{y}}] p(y,x)p(\widetilde{y},\widetilde{x}) + \alpha m^2 \sum_y P[N_{t-1,y,y}] p(y,x)p(y,\widetilde{x}) \\ &\quad + (m^{(2)} - m)\delta_{x,\widetilde{x}} m^{t-1} p_t(0,x) \\ &= \sum_{y,\widetilde{y}} P[N_{t-1,y,\widetilde{y}}] a(y,\widetilde{y}) p(y,x)p(\widetilde{y},\widetilde{x}) + b_t(x,\widetilde{x}), \end{aligned}$$

where

$$a(y,\widetilde{y}) = m^2\alpha^{1\{y=\widetilde{y}\}}, \text{ and } b_t(x,\widetilde{x}) = cm^t\delta_{x,\widetilde{x}}p_t(0,x).$$

By Lemma 2.1.2 below, applied to the Markov chain $(S,\widetilde{S})$, we get

$$P[N_{T,x,\widetilde{x}}] = c\sum_{t=0}^{T-1} m^{T+t}P^{x,\widetilde{x}}_{S,\widetilde{S}}\left[p_{T-t}(0,S_t)\alpha^{\sum_{u=1}^t 1\{S_u=\widetilde{S}_u\}} : S_t = \widetilde{S}_t\right].$$

It is now easy to see that the above identity is the same as what we want. $\square$

Lemma 2.1.2 *Let $S = (S_t)_{t\in\mathbb{N}}$ be a Markov chain with the state space Γ. Suppose that φ_t, a_t, b_t ($t \in \mathbb{N}$) are functions on Γ such that*

$$\varphi_t(x) = P^x_S[a_t(S_1)\varphi_{t-1}(S_1)] + b_t(x), \quad x \in \Gamma, \ t \geq 1, \tag{2.1}$$

where P^x_S denotes the law of S, conditioned to start from $x \in \Gamma$. Then,

$$\varphi_T(x) = P^x_S\left[\varphi_0(S_T)\prod_{u=1}^T a_{T-u+1}(S_u) + \sum_{t=0}^{T-1} b_{T-t}(S_t)\prod_{u=1}^t a_{T-u+1}(S_u)\right], \quad x \in \Gamma, \ T \geq 1. \tag{2.2}$$

((2.1) and (2.2) are discrete analogues of the parabolic Schrödinger equation and its Feynman-Kac representation.)

Proof: Straightforward by induction on T. $\square$

Lemma 2.1.3 *For functions $f, \widetilde{f}$ on $\mathbb{Z}^d$,*

$$\sum_{x,\widetilde{x}\in\mathbb{Z}^d} P[N_{T,x}N_{T,\widetilde{x}}]f(x)\widetilde{f}(\widetilde{x})$$

$$= m^T P_S[f(S_T)\widetilde{f}(S_T)] + cm^T\sum_{t=0}^{T-1} m^t P^{0,0}_{S,\widetilde{S}}\left[\alpha^{\sum_{u=1}^t 1\{S_t-S_u=\widetilde{S}_t-\widetilde{S}_u\}} f(-S_T)\widetilde{f}(S_t-\widetilde{S}_t-S_T)\right],$$

where $\alpha = \frac{Q[m^2_{t,x}]}{m^2}$ and $c = \dfrac{m^{(2)}}{m} - 1$.

Proof: For $0 \leq t \leq T$, we compute

$$\begin{aligned}
I_t &\stackrel{\text{def.}}{=} \sum_{x,\widetilde{x}\in\mathbb{Z}^d} P^{x,\widetilde{x}}_{S,\widetilde{S}}\left[\alpha^{\sum_{u=1}^t 1\{S_u=\widetilde{S}_u\}} : S_t = \widetilde{S}_t, \ S_T = 0\right] f(x)\widetilde{f}(\widetilde{x}) \\
&= \sum_{x,\widetilde{x}\in\mathbb{Z}^d} P^{0,0}_{S,\widetilde{S}}\left[\alpha^{\sum_{u=1}^t 1\{S_u-\widetilde{S}_u=\widetilde{x}-x\}} : S_t - \widetilde{S}_t = \widetilde{x} - x, \ S_T = -x\right] f(x)\widetilde{f}(\widetilde{x}) \\
&= \sum_{x,\widetilde{x}\in\mathbb{Z}^d} P^{0,0}_{S,\widetilde{S}}\left[\alpha^{\sum_{u=1}^t 1\{S_u-\widetilde{S}_u=S_t-\widetilde{S}_t\}} : S_t - \widetilde{S}_t - S_T = \widetilde{x}, \ S_T = -x\right] f(x)\widetilde{f}(\widetilde{x}) \\
&= P^{0,0}_{S,\widetilde{S}}\left[\alpha^{\sum_{u=1}^t 1\{S_t-S_u=\widetilde{S}_t-\widetilde{S}_u\}} f(-S_T)\widetilde{f}(S_t-\widetilde{S}_t-S_T)\right].
\end{aligned}$$

By Lemma 2.1.1, the left-hand-side of the desired identity equals

$$m^T P_S[f(S_T)\widetilde{f}(S_T)] + cm^T\sum_{t=0}^{T-1} m^t I_t.$$

$\square$

Lemma 2.1.4 *Suppose that $1 \leq \alpha < 1/\pi_d$. Then, for $f, \widetilde{f} \in C_b(\mathbb{R}^d)$,*

$$\lim_{t\to\infty} P^{0,0}_{S,\widetilde{S}}\left[\alpha^{\sum_{u=0}^{t-1} 1\{S_u=\widetilde{S}_u\}} f(t^{-1/2}S_t)\widetilde{f}(t^{-1/2}\widetilde{S}_t)\right] = P^{0,0}_{S,\widetilde{S}}\left[\alpha^{\sum_{u=0}^{\infty} 1\{S_u=\widetilde{S}_u\}}\right]\left(\int_{\mathbb{R}^d} fg_1\right)\left(\int_{\mathbb{R}^d}\widetilde{f}g_1\right).$$

Proof: This lemma is shown in the proof of [6, Theorem 4.2]. $\square$

2.2 Proof of Theorem 1.2.1

(a) $\Longleftrightarrow$ (b): It follows from Lemma 2.1.3 that

$$P[\overline{N}_t^2] = m^{-T} + cm^{-T} \sum_{t=0}^{T-1} m^t P^{0,0}_{S,\widetilde{S}} \left[\alpha^{\sum_{u=1}^t 1\{S_u = \widetilde{S}_u\}} \right]$$

and hence that

$$\sup_{t \geq 0} P[\overline{N}_t^2] = \lim_{t \to \infty} P[\overline{N}_t^2] = cP^{0,0}_{S,\widetilde{S}} \left[\alpha^{\sum_{u=1}^\infty 1\{S_u = \widetilde{S}_u\}} \right] = cP_S \left[\alpha^{\sum_{u=1}^\infty 1\{S_{2u}=0\}} \right].$$

The right-hand-side is finite if and only if $\alpha < 1/\pi_d$, since $\sum_{u=1}^\infty 1\{S_{2u} = 0\}$ is geometrically distributed with the success probability π_d.

(a) $\Rightarrow$ (c): By a standard approximation arguments, we may assume that $f \in C_{b,u}(\mathbb{R}^d)$, where $C_{b,u}(\mathbb{R}^d)$ denotes the set of bounded, uniformly continuous functions. Since (a) implies (b), it is enough to prove that, as $T \nearrow \infty$,

$$X_T \stackrel{\text{def.}}{=} \sum_{x \in \mathbb{Z}^d} \overline{N}_{T,x} f\left(T^{-1/2}x\right) \longrightarrow 0 \quad \text{in } \mathbb{L}^2(P)$$

for $f \in C_{b,u}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} f g_1 = 0$. By Lemma 2.1.3,

$$P[X_T^2] = \sum_{x,\widetilde{x} \in \mathbb{Z}^d} P[\overline{N}_{T,x}\overline{N}_{T,\widetilde{x}}] f_T(x) f_T(\widetilde{x}) = m^{-T} P_S[f_T(S_T)^2] + cm^{-T} \sum_{t=0}^{T-1} m^t \gamma_{t,T},$$

where $f_T(x) = f(T^{-1/2}x)$ and

$$\gamma_{t,T} = P^{0,0}_{S,\widetilde{S}} \left[\alpha^{\sum_{u=1}^t 1\{S_t - S_u = \widetilde{S}_t - \widetilde{S}_u\}} f_T(-S_T) f_T(S_t - \widetilde{S}_t - S_T) \right].$$

Since

$$\lim_{T \to \infty} m^{-T} \sum_{1 \leq t \leq T - \ln T} m^t \gamma_{t,T} = 0,$$

it is enough to show that

$$(*) \qquad \lim_{T \to \infty} \sup\{\gamma_{t,T} \; ; \; T - \ln T \leq t < T\} = 0.$$

For $T - \ln T \leq t < T$, both

$$|T^{-1/2} S_T - t^{-1/2} S_t| \text{ and } |T^{-1/2}(S_t - \widetilde{S}_t - S_T) + t^{-1/2}\widetilde{S}_t|$$

are bounded by a constant multiple of $T^{-1/2} \ln T$. Since f is uniformly continuous and $\alpha < 1/\pi_d$, we have

$$\begin{aligned} \gamma_{t,T} &= P^{0,0}_{S,\widetilde{S}} \left[\alpha^{\sum_{u=1}^t 1\{S_t - S_u = \widetilde{S}_t - \widetilde{S}_u\}} f_t(-S_t) f_t(-\widetilde{S}_t) \right] + \varepsilon_T \\ &= P^{0,0}_{S,\widetilde{S}} \left[\alpha^{\sum_{u=0}^{t-1} 1\{S_u = \widetilde{S}_u\}} f_t(-S_t) f_t(-\widetilde{S}_t) \right] + \varepsilon_T \end{aligned}$$

with some $\varepsilon_T \to 0$. Here, on the second line, we have used that $(S_u)_{u=0}^t \stackrel{\text{law}}{=} (S_t - S_{t-u})_{u=0}^t$. This, together with Lemma 2.1.4, implies $(*)$.

(c) $\Rightarrow$ (b): Obvious. $\square$

2.3 Proof of Proposition 1.2.3

It follows from the assumptions and Theorem 1.2.1 that $P(\overline{N}_\infty > 0) > 0$. Note that

$$\frac{1}{N_T^2}\sum_{x\in\mathbb{Z}^d} N_{T,x}^2 = \frac{1}{\overline{N}_T^2}\sum_{x\in\mathbb{Z}^d} \overline{N}_{T,x}^2$$

and that $\lim_{T\to\infty}\overline{N}_T = \overline{N}_\infty > 0$, $P(\,\cdot\,|\overline{N}_\infty > 0)$-a.s. Therefore, it is enough to prove that

$$\sum_{x\in\mathbb{Z}^d} P(\overline{N}_{T,x}^2|\overline{N}_\infty > 0) = O(T^{-d/2}).$$

In fact, we will show that

(1) $\quad \sum_{x\in\mathbb{Z}^d} P(\overline{N}_{T,x}^2) = O(T^{-d/2})$.

Since $\alpha < 1/\pi_d$, we have

$$0 < \inf_{\mathbb{Z}^d}\Phi \le \sup_{\mathbb{Z}^d}\Phi < \infty \ \text{ for } \ \Phi(x) = P_S^x[\alpha^{\sum_{u=1}^\infty 1\{S_{2u}=0\}}].$$

Then, it follows from Lemma 2.3.1 below that

(2) $\quad \sup_{x\in\mathbb{Z}^d} P_S^0\left[\alpha^{\sum_{u=1}^t 1\{S_{2u}=0\}} : S_{2t} = x\right] = O(t^{-d/2}),\ t\nearrow\infty.$

By Lemma 2.1.1,

$$\begin{aligned} P(\overline{N}_{T,x}^2) &= m^{-T}P_S(S_T = x) + cm^{-T}\sum_{t=0}^{T-1} m^t P_{S,\widetilde{S}}^{x,x}\left[\alpha^{\sum_{u=1}^t 1\{S_u=\widetilde{S}_u\}} : S_t = \widetilde{S}_t,\ S_T = 0\right] \\ &= m^{-T}P_S(S_T = x) + cm^{-T}\sum_{t=0}^{T-1} m^t P_{S,\widetilde{S}}^{0,0}\left[\alpha^{\sum_{u=1}^t 1\{S_u=\widetilde{S}_u\}} : S_t = \widetilde{S}_t,\ S_T = -x\right]. \end{aligned}$$

Hence, (1) can be seen via (2) as follows:

$$\begin{aligned} \sum_{x\in\mathbb{Z}^d} P(\overline{N}_{T,x}^2) &= m^{-T} + m^{-T}\sum_{t=0}^{T-1} m^t P_{S,\widetilde{S}}^{0,0}\left[\alpha^{\sum_{u=1}^t 1\{S_u=\widetilde{S}_u\}} : S_t = \widetilde{S}_t\right] \\ &= m^{-T} + m^{-T}\sum_{t=0}^{T-1} m^t P_S^0\left[\alpha^{\sum_{u=1}^t 1\{S_{2u}=0\}} : S_{2t} = 0\right] = O(T^{-d/2}). \end{aligned}$$

□

Lemma 2.3.1 *Suppose that $b : \mathbb{Z}^d \to \mathbb{R}$ $(d \ge 3)$ is bounded and and that*

$$0 < \inf_{\mathbb{Z}^d}\Phi \le \sup_{\mathbb{Z}^d}\Phi < \infty \ \textit{ with } \ \Phi(x) = P_S^x\left[\exp\left(\sum_{t=0}^\infty b(S_t)\right)\right].$$

Then, as $T\nearrow\infty$,

$$\sup_{x,y\in\mathbb{Z}^d} P_S^x\left[\exp\left(\sum_{t=0}^{T-1} b(S_t)\right) : S_T = y\right] = O(T^{-d/2}).$$

Proof: Here is a receipe to get a Nash type estimate for Schrödinger semigroup, which is similar to [10, Lemma 3.1.3], [19, Lemma 3.3]. We define

$$\widetilde{p}(x,y) = \frac{1}{\Phi(x)} e^{b(x)} p(x,y) \Phi(y), \quad \widetilde{m}(x) = e^{-b(x)} \Phi^2(x).$$

Then, it is easy to see that $\widetilde{p}$ is a transition probability of an $\widetilde{m}$-symmetric Markov chain on $\mathbb{Z}^d$. Moreover, we have by the assumption that

(1) $$0 < \inf_{\mathbb{Z}^d} \widetilde{m} \le \sup_{\mathbb{Z}^d} \widetilde{m} < \infty \text{ ("strong reversibility" [20, page 27])},$$

(2) $$\left(\sum_{x \in A} \widetilde{m}(x) \right)^{\frac{d-1}{d}} \le \kappa \sum_{x \in A, y \notin A} \widetilde{m}(x) \widetilde{p}(x,y), \text{ for any finite } A \subset \mathbb{Z}^d,$$

where the constant κ is independent of A ("d-isoperimetric inequality" [20, page 40]),

(3) $$\sum_{\substack{y \in \mathbb{Z}^d \\ |x-y| \le r}} \widetilde{m}(y) \ge \varepsilon r^d \text{ for any } r \in \mathbb{N} \text{ and } x \in \mathbb{Z}^d,$$

where the constant $\varepsilon > 0$ is independent of r and x.

Then, by [20, page 148, Corollary 14.5], these imply that

(4) $$\sup_{x,y \in \mathbb{Z}^d} \widetilde{p}_T(x,y) = O(T^{-d/2}) \text{ as } T \nearrow \infty,$$

where $\widetilde{p}_T$ is the T step transition function obtained from $\widetilde{p}$.

Since

$$P_S^x \left[\exp \left(\sum_{t=0}^{T-1} b(S_t) \right) : S_T = y \right] = \Phi(x) \widetilde{p}_T(x,y) \frac{1}{\Phi(y)},$$

the lemma follows from (4). □

3 Relation to directed polymers in random environment

We now relate the BRWRE with directed polymers in random environment.

3.1 Directed polymers in random environment (DPRE)

• *The random environment:* $\eta = \{\eta_{t,x} : (x,t) \in \mathbb{Z}^d \times \mathbb{N}\}$ is a sequence of r.v.'s which are real valued, non-constant, and i.i.d. r.v.'s defined on a probability space $(\Omega_\eta, \mathcal{F}_\eta, Q)$ such that

$$Q[\exp(\beta \eta_{t,x})] < \infty \quad \text{for all } \beta \in \mathbb{R}. \tag{3.1}$$

We define

$$\lambda(\beta) = \ln Q[\exp(\beta \eta_{t,x})]. \tag{3.2}$$

• *The polymer measure:* For any $T \in \mathbb{N}^*$, define the probability measure μ_T on the path space $(\Omega_S, \mathcal{F}_S)$ by

$$d\mu_T = \frac{1}{Z_T} \exp(\beta H_T)\, dP_S, \tag{3.3}$$

where $\beta > 0$ is a parameter (the inverse temperature),

$$H_T = \sum_{t=0}^{T-1} \eta_{t,S_t} \quad \text{and} \quad Z_T = P_S \left[\exp \left(\beta H_T \right) \right] \tag{3.4}$$

are the Hamiltonian and the normalizing constant (the partition function).
Define the *normalized partition function* by

$$\overline{Z}_T = Z_T \exp(-T\lambda(\beta)) \ , \ \ T \geq 1. \tag{3.5}$$

This is a positive mean-one $(\mathcal{F}_{\eta,T})$-martingale on $(\Omega_\eta, \mathcal{F}_\eta, Q)$ with $\mathcal{F}_{\eta,0} = \{\emptyset, \Omega_\eta\}$ and

$$\mathcal{F}_{\eta,T} = \sigma(\eta(\cdot, s) \ ; \ s \leq T-1), \quad T \geq 1.$$

By the martingale convergence theorem, the limit

$$\overline{Z}_\infty = \lim_{T\to\infty} \overline{Z}_T \tag{3.6}$$

exists Q-a.s. Moreover, there are only two possibilities for the positivity of the limit;

$$Q\{\overline{Z}_\infty > 0\} = 1 \ , \tag{3.7}$$

or

$$Q\{\overline{Z}_\infty = 0\} = 1 \ . \tag{3.8}$$

Indeed, the event $\{\overline{Z}_\infty = 0\}$ is in the tail σ-field:

$$\bigcap_{t\geq 1} \sigma[\eta(\cdot, s) \ ; \ s \geq t] \ .$$

By Kolmogorov's zero-one law, every event in the tail σ-field has probability 0 or 1.
The above contrasting situations (3.7) and (3.8) will be called the **weak disorder** phase and the **strong disorder** phase, respectively.

3.2 BRWRE and its associated DPRE

Suppose that we are given an environment $\eta_{t,x}$ for the directed polymer. We can then associate an environment, i.e., an i.i.d. random offspring distribution $q_{t,x}$ for the BRWRE so that

$$e^{\beta\eta_{t,x}} = m_{t,x} = \sum_{k\in\mathbb{N}} k q_{t,x}(k) \tag{3.9}$$

holds. Among many ways to do so, let us take:

$$q_{t,x}(k) = e^{-m_{t,x}} \frac{m_{t,x}^k}{k!}. \tag{3.10}$$

Then, by Lemma 1.3.1,

$$P^q[N_{t,x}] = P_S[\exp(\beta H_T) : S_T = x] \ \text{ and } \ P^q[N_T] = Z_T. \tag{3.11}$$

These imply

$$P^q[\overline{N}_{T,x}] = P_S[\exp(\beta H_T - \lambda(\beta)T) : S_T = x], \ \ P^q[\overline{N}_T] = \overline{Z}_T$$

and

$$\mu_T(S_T = x) = \frac{P^q[N_{t,x}]}{P^q[N_T]} = \frac{P^q[\overline{N}_{t,x}]}{P^q[\overline{N}_T]}.$$

Since μ_T and $\overline{Z}_T$ are invariant under constant addition to $\eta_{t,x}$, we may assume that

$$m = Q[m_{t,x}] = Q[\exp(\beta\eta_{t,x})] > 1$$

without loss of generality. Moreover, by (3.10) and our integrability assumption (3.1) on $\eta_{t,x}$,

$$\sum_{k \in \mathbb{N}} k^2 q_{t,x}(k) = m_{t,x} + m_{t,x}^2 \in \mathbb{L}^1(Q).$$

Therefore, Theorem 1.2.1 implies the following central limit theorem for DPRE, which is a weaker version of the results obtained in [4, 17, 12].

Corollary 3.2.1 *If $d \geq 3$ and $\lambda(2\beta) - 2\lambda(\beta) < \ln(1/\pi_d)$, then*

$$\lim_{T \to \infty} \mu_T[f(T^{-1/2} S_T)] = \int_{\mathbb{R}^d} f g_1 \quad \text{in } Q\text{-probability for any } f \in C_b(\mathbb{R}^d), \tag{3.12}$$

3.3 Phase transitions of BRWRE and DPRE

As we mentioned before, BRWRE undergoes the following phase transition. When the randomness of the offspring distribution is well moderated by that of the random walk (as in Corollary 1.2.2, Proposition 1.2.3), the growth of the total population is of the same order as its expectation with strictly positive probability. When, on the other hand, the randomness of the environment dominates, the total population grows strictly slower than its expectation almost surely. We now relate this phase transition with that for DPRE.

Proposition 3.3.1 *Suppose that an environment $\eta_{t,x}$ for the DPRE and an offspring distribution $q_{t,x}$ for the BRWRE are related so that (3.9) holds. Then,*

(a) $P[\overline{N}_\infty] \leq Q[\overline{Z}_\infty]$. *In particular, $Q(\overline{Z}_\infty = 0) = 1$ (strong disorder for DPRE) implies $P(\overline{N}_\infty = 0) = 1$ (the total population grows strictly slower than its expectation almost surely).*

(b) *The converse to (a) is not true.*

Proof: (a): We have:

$$P[\overline{N}_\infty] = \int Q(dq) P^q[\overline{N}_\infty] \leq \int Q(dq) \lim_t P^q[\overline{N}_t] = Q[\overline{Z}_\infty],$$

where the inequality follows from Fatou's lemma.
(b) As is mentioned in [3, Theorem 4], It follows from a comparison with the Galton-Watson model that $P(N_\infty = 0) = 1$ and hence $P(\overline{N}_\infty = 0) = 1$ as soon as $m \leq 1$. We have always $m \geq 1$ when $\eta_{t,x}$ is a negative r.v. Meanwhile, the polymer is in weak disorder phase for $d \geq 3$ and $\beta > 0$ small enough. □

By Proposition 3.3.1, we can translate the results from DPRE [7, 9] to the following observation for BRWRE:

Corollary 3.3.2 *Suppose one of the following conditions:*

(a1) $d = 1,\ Q(m_{t,x} = m) \neq 1$.

(a2) $d = 2,\ Q(m_{t,x} = m) \neq 1$.

(a3) $d \geq 3,\ Q\left[\frac{m_{t,x}}{m} \ln \frac{m_{t,x}}{m}\right] > \ln(2d)$.

Then, $P(\overline{N}_\infty = 0) = 1$. Moreover, in cases (a1) and (a3), there exists a non-random number $c > 0$ such that

$$\overline{\lim_t} \frac{\ln \overline{N}_t}{t} < -c, \quad a.s. \tag{3.13}$$

Corollary 3.3.2 says that the total population grows strictly slower than its expectation almost surely, in low dimensions or in "random enough" environment. The result is in contrast with the non-random environment case, where $P(\overline{N}_\infty = 0) = 1$ only for offspring distributions with very heavy tail, more precisely, if and only if $P[K^\nu_{t,x} \ln K^\nu_{t,x}] = \infty$ [1, page 24, Theorem 1]. Here, we can have $P(\overline{N}_\infty = 0) = 1$ even when $K^\nu_{t,x}$ is bounded. Also, (3.13) is in sharp contrast with the non-random environment case, where it is well known –see e.g., [1, page 30, Theorem 3] –that

$$\{N_\infty > 0\} \stackrel{\text{a.s.}}{=} \{\lim_t \frac{\ln \overline{N}_t}{t} = 0\} \quad \text{whenever } m > 1.$$

Corollary 3.3.2 is also stated in [3, Theorem 4], however without (3.13).

We have discussed the relation between BRWRE and DPRE. We remark that the branching Brownian motions in random environment recently considered by Shiozawa [16] and Brownian directed polymers [11] are in similar relation. Therefore, by the results in [11], Proposition 3.3.1 and Corollary 3.3.2 (except (3.13) in case (a1)) can be extended to branching Brownian motions in random environment.

4 Survival and extinction

We now close this article with a brief discussion on survival/extinction for this model, i.e., $\mathbf{e} \stackrel{\text{def}}{=} P(N_\infty = 0) < 1$ or $= 1$. By standard computations of generating function, we see that $\mathbf{e}^{\text{GW}} \le \mathbf{e} \le \mathbf{e}^{\text{SW}}$, where $\mathbf{e}^{\text{GW}}$ and $\mathbf{e}^{\text{SW}}$ stands respectively for extinction probabilities for Galton-Watson model with offspring distribution $Q[q_{t,x}(\,\cdot\,)]$ and Smith-Wilkinson model [1, 18]. On the other hand, by using oriented percolation, one can construct examples for $\mathbf{e}^{\text{GW}} < \mathbf{e} = 1$ and for $\mathbf{e} < \mathbf{e}^{\text{SW}} = 1$ [13].

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